C3 S1S

1. Given that

$$\tan\theta° = p, \text{ where } p \text{ is a constant, } p \neq \pm 1$$

use standard trigonometric identities, to find in terms of p,

(a) $\tan 2\theta°$ **(2)**

(b) $\cos\theta°$ **(2)**

(c) $\cot(\theta - 45)°$ **(2)**

Write each answer in its simplest form.

a) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{2p}{1-p^2}$

b) $\dfrac{\sin^2\theta}{\cos^2\theta} + \dfrac{\cos^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta} \Rightarrow \tan^2\theta + 1 = \dfrac{1}{\cos^2\theta}$

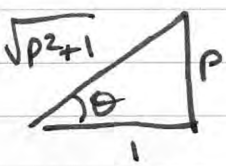
$\therefore \cos^2\theta = \dfrac{1}{\tan^2\theta + 1} \Rightarrow \cos\theta = \sqrt{\dfrac{1}{\tan^2\theta + 1}}$

$\therefore \cos\theta = \dfrac{1}{\sqrt{p^2+1}}$

c) $\cot(\theta - 45) = \dfrac{1}{\tan(\theta - 45)} = \dfrac{1 + \tan\theta\tan 45}{\tan\theta - \tan 45}$

$= \dfrac{1+\tan\theta}{\tan\theta - 1} = \dfrac{1+p}{p-1}$

alt b) $\tan\theta = \dfrac{p}{1}$



$\therefore \cos\theta = \dfrac{1}{\sqrt{p^2+1}}$

2. Given that

$$f(x) = 2e^x - 5, \quad x \in \mathbb{R}$$

(a) sketch, on separate diagrams, the curve with equation

(i) $y = f(x)$

(ii) $y = |f(x)|$

On each diagram, show the coordinates of each point at which the curve meets or cuts the axes.

On each diagram state the equation of the asymptote.

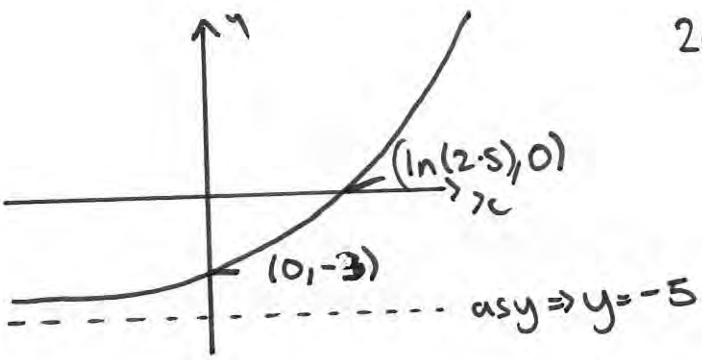
(6)

(b) Deduce the set of values of x for which $f(x) = |f(x)|$

(1)

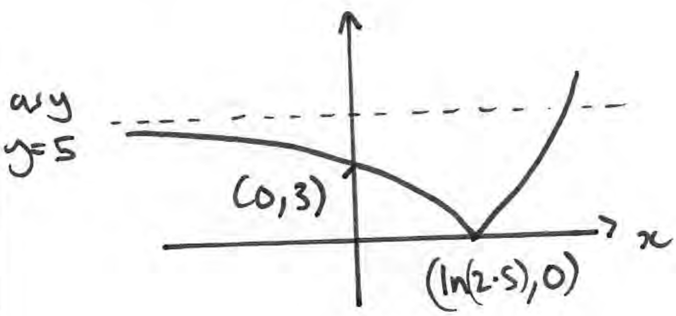
(c) Find the exact solutions of the equation $|f(x)| = 2$

(3)



$2e^x = 5 \Rightarrow e^x = 2.5$

$x = \ln 2.5$



b) $x \geqslant \ln 2.5$

c) $|f(x)| = 2 \Rightarrow 2e^x - 5 = 2$

$2e^x = 7$

$e^x = 3.5$

$x = \ln 3.5$

$2e^x - 5 = -2$

$2e^x = 3$

$e^x = 1.5$

$x = \ln 1.5$

3. $$g(\theta) = 4\cos 2\theta + 2\sin 2\theta$$

Given that $g(\theta) = R\cos(2\theta - \alpha)$, where $R > 0$ and $0 < \alpha < 90°$,

(a) find the exact value of R and the value of α to 2 decimal places. **(3)**

(b) Hence solve, for $-90° < \theta < 90°$,

$$4\cos 2\theta + 2\sin 2\theta = 1$$

giving your answers to one decimal place. **(5)**

Given that k is a constant and the equation $g(\theta) = k$ has no solutions,

(c) state the range of possible values of k. **(2)**

a) $R\cos(2\theta - \alpha) = R\cos 2\theta \cos\alpha + R\sin 2\theta \sin\alpha$

$4\cos 2\theta + 2\sin 2\theta$

$\dfrac{R\sin\alpha = 2}{R\cos\alpha = 4} \Rightarrow \tan\alpha = \dfrac{1}{2} \quad \therefore \alpha = 26.57$

$R^2 = 2^2 + 4^2 \Rightarrow R = \sqrt{20} = 2\sqrt{5}$

$4\cos 2\theta + 2\sin 2\theta = 2\sqrt{5}\cos(2\theta - 26.57)$

b) $2\sqrt{5}\cos(2\theta - 26.57) = 1$

$\Rightarrow \cos(2\theta - 26.57) = \dfrac{1}{2\sqrt{5}}$

$\Rightarrow 2\theta - 26.57 = \cos^{-1}\left(\dfrac{1}{2\sqrt{5}}\right) = 77.079, -77.079$

$\Rightarrow 2\theta = 103.64 ; -50.51 \quad \therefore \theta = -\dfrac{25.3}{2} ; \dfrac{51.8°}{2}$

c) $g(\theta) = 2\sqrt{5}\cos(2\theta - 26.57) = k$

no solution if $\dfrac{k}{2\sqrt{5}} > 1$ or $\dfrac{k}{2\sqrt{5}} < -1$

$k > 2\sqrt{5}$ or $k < -2\sqrt{5}$

4. Water is being heated in an electric kettle. The temperature, θ °C, of the water t seconds after the kettle is switched on, is modelled by the equation

$$\theta = 120 - 100e^{-\lambda t}, \qquad 0 \leqslant t \leqslant T$$

(a) State the value of θ when $t = 0$

(1)

Given that the temperature of the water in the kettle is 70 °C when $t = 40$,

(b) find the exact value of λ, giving your answer in the form $\frac{\ln a}{b}$, where a and b are integers.

(4)

When $t = T$, the temperature of the water reaches 100 °C and the kettle switches off.

(c) Calculate the value of T to the nearest whole number.

(2)

a) $t = 0 \quad \theta = 120 - 100(1) = \underline{20°}$

b) $70 = 120 - 100e^{-40\lambda}$

$\Rightarrow -50 = -100e^{-40\lambda} \Rightarrow e^{-40\lambda} = \frac{1}{2} \Rightarrow -40\lambda = \ln(\frac{1}{2})$

$\Rightarrow \cancel{-}40\lambda = \cancel{-}\ln 2 \Rightarrow \lambda = \frac{1}{40}\ln 2 \qquad a = 2, b = 40$

c) $100 = 120 - 100e^{-\left(\frac{\ln 2}{40}\right)T}$

$\Rightarrow \cancel{-}20 = \cancel{-}100e^{\left(-\frac{\ln 2}{40}\right)T} \Rightarrow e^{\left(-\frac{\ln 2}{40}\right)T} = \frac{1}{5}$

$-\left(\frac{\ln 2}{40}\right)T = \ln\left(\frac{1}{5}\right) \Rightarrow \cancel{-}\left(\frac{\ln 2}{40}\right)T = \cancel{-}\ln 5$

$\Rightarrow (\ln 2)T = 40\ln 5$

$\Rightarrow T = \frac{40\ln 5}{\ln 2} = 92.877$

$\therefore \underline{T \simeq 93}$

5. The point P lies on the curve with equation

$$x = (4y - \sin 2y)^2$$

Given that P has (x, y) coordinates $\left(p, \frac{\pi}{2}\right)$, where p is a constant,

(a) find the exact value of p.

(1)

The tangent to the curve at P cuts the y-axis at the point A.

(b) Use calculus to find the coordinates of A.

(6)

a) $x = (4y - \sin 2y)^2 \quad y = \frac{\pi}{2} \quad x = (4y - \cancel{\sin \pi})^2$

$x = \left(4\frac{\pi}{2}\right)^2 = \frac{4\pi^2}{2}$

b) $\frac{dx}{dy} = 2(4y - \sin 2y)^1 \times (4 - 2\cos 2y)$

$\left.\frac{dx}{dy}\right|_{y=\frac{\pi}{2}} = 2\left(4\frac{\pi}{2} - 0\right)^1 \times (4 - (-2)) = 24\pi$

$\therefore M_t = \frac{1}{20\pi} \Rightarrow y - \frac{\pi}{2} = \frac{1}{24\pi}(x - 4\pi^2)$

$24\pi y - 12\pi^2 = x - 4\pi^2$

$24\pi y = x + 8\pi^2$

Cuts y when $x = 0 \Rightarrow 24\pi y = 8\pi^2$

$y = \frac{\pi}{3} \quad \left(0, \frac{\pi}{3}\right)$

6.

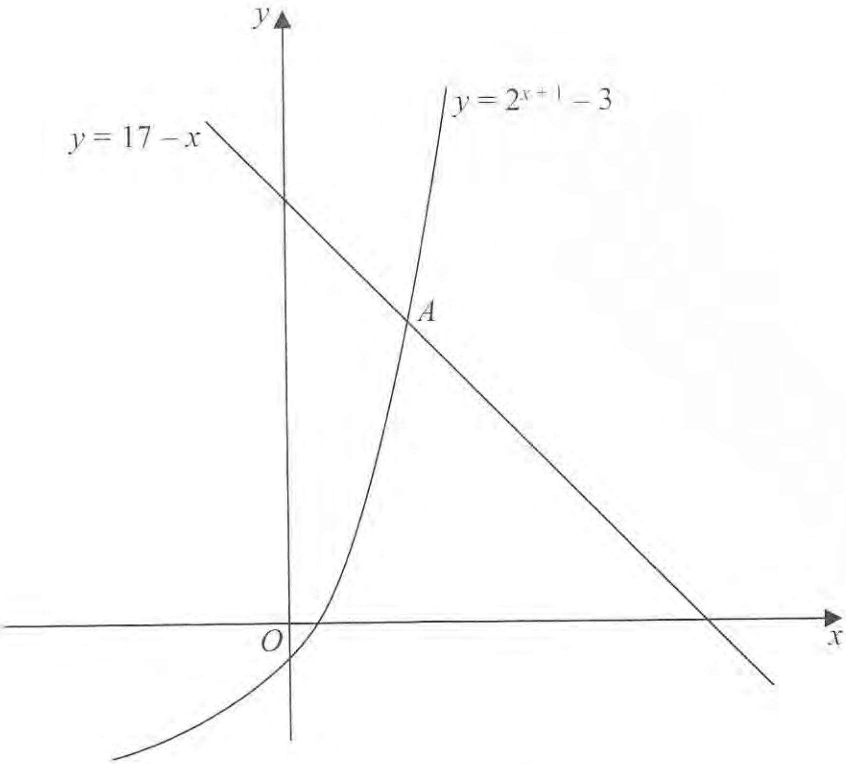


Figure 1

Figure 1 is a sketch showing part of the curve with equation $y = 2^{x+1} - 3$ and part of the line with equation $y = 17 - x$.

The curve and the line intersect at the point A.

(a) Show that the x coordinate of A satisfies the equation

$$x = \frac{\ln(20 - x)}{\ln 2} - 1$$

(3)

(b) Use the iterative formula

$$x_{n+1} = \frac{\ln(20 - x_n)}{\ln 2} - 1, \quad x_0 = 3$$

to calculate the values of x_1, x_2 and x_3, giving your answers to 3 decimal places.

(3)

(c) Use your answer to part (b) to deduce the coordinates of the point A, giving your answers to one decimal place.

(2)

a) $17 - x = 2^{x+1} - 3 \quad \Rightarrow 20 - x = 2^{x+1}$

$\Rightarrow \ln(20-x) = \ln 2^{(x+1)}$

$\Rightarrow \ln(20-x) = (x+1)\ln 2$

$\Rightarrow x+1 = \dfrac{\ln(20-x)}{\ln 2}$

$\therefore x = \dfrac{\ln(20-x)}{\ln 2} - 1$

b) $x_0 = 3$

$x_1 = 3.087$

$x_2 = 3.080$

$x_3 = 3.081$

$x_n \rightarrow 3.0806 \ldots$

$x \rightarrow 3.081 \ldots$

$y \rightarrow 13.919 \ldots$

$A(3.1, 13.9)$

7.

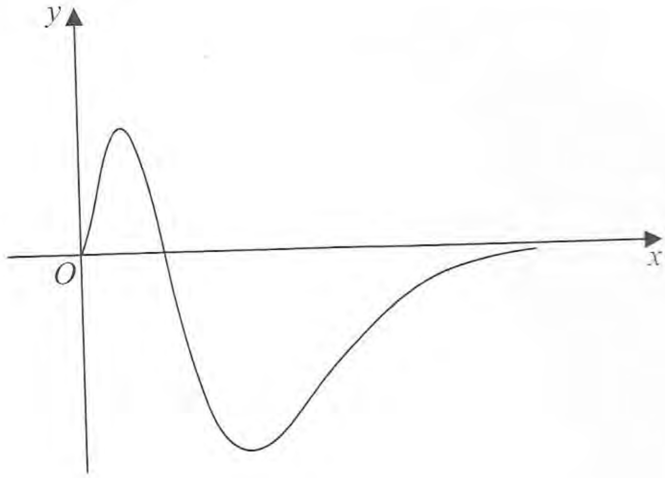


Figure 2

Figure 2 shows a sketch of part of the curve with equation

$$g(x) = x^2(1-x)e^{-2x}, \quad x \geqslant 0$$

(a) Show that $g'(x) = f(x)e^{-2x}$, where $f(x)$ is a cubic function to be found. **(3)**

(b) Hence find the range of g. **(6)**

(c) State a reason why the function $g^{-1}(x)$ does not exist. **(1)**

$g(x) = (x^2 - x^3)e^{-2x}$ $\quad u = x^2 - x^3 \quad v = e^{-2x}$

$u' = 2x - 3x^2 \quad v' = -2e^{-2x}$

$\therefore g'(x) = (2x - 3x^2)e^{-2x}$ $\quad vu' + uv'$

$-2(x^2 - x^3)e^{-2x}$

$= (2x - 3x^2 - 2x^2 + 2x^3)e^{-2x}$

$= (2x^3 - 5x^2 + 2x)e^{-2x}$

b) at TP $g'(x) = 0 \Rightarrow x(2x^2 - 5x + 2) = 0$

$2x^2 - 5x + 2 = 0 \Rightarrow (2x-1)(x-2)$

$x = \frac{1}{2} \quad x = 2$

$x = 2 \Rightarrow g(2) = 4(1-2)e^{-4} = -4e^{-4}$

$x = \frac{1}{2} \Rightarrow g(\frac{1}{2}) = \frac{1}{4}(1-\frac{1}{2})e^{-1} = \frac{1}{8}e^{-1}$

$$-4e^{-4} \le g(x) \le \frac{1}{8}e^{-1}$$

8. (a) Prove that

$$\sec 2A + \tan 2A \equiv \frac{\cos A + \sin A}{\cos A - \sin A}, \qquad A \neq \frac{(2n+1)\pi}{4},\ n \in \mathbb{Z}$$

(5)

(b) Hence solve, for $0 \leqslant \theta < 2\pi$,

$$\sec 2\theta + \tan 2\theta = \frac{1}{2}$$

Give your answers to 3 decimal places.

(4)

$$\frac{(\cos A + \sin A) \times (\cos A + \sin A)}{(\cos A - \sin A) \times (\cos A + \sin A)}$$

$$= \frac{\cos^2 A + \sin^2 A + 2\sin A\cos A}{\cos^2 A - \sin^2 A}$$

$$= \frac{2\sin A\cos A + 1}{\cos 2A} = \frac{\sin 2A + 1}{\cos 2A}$$

$$= \frac{1}{\cos 2A} + \frac{\sin 2A}{\cos 2A} = \sec 2A + \tan 2A \quad \#$$

b) $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \dfrac{1}{2} \Rightarrow 2\cos\theta + 2\sin\theta = \cos\theta - \sin\theta$

$\Rightarrow 3\sin\theta = -\cos\theta$

$\therefore \tan\theta = -\frac{1}{3}$

$\theta = -0.32, 2.8198, 5.961$

$\therefore \theta = 2.820 \ ; 5.961$

9. Given that k is a **negative** constant and that the function $f(x)$ is defined by

$$f(x) = 2 - \frac{(x-5k)(x-k)}{x^2 - 3kx + 2k^2}, \quad x \geqslant 0$$

(a) show that $f(x) = \dfrac{x+k}{x-2k}$

(3)

(b) Hence find $f'(x)$, giving your answer in its simplest form.

(3)

(c) State, with a reason, whether $f(x)$ is an increasing or a decreasing function.

Justify your answer.

(2)

a) $f(x) = 2 - \dfrac{(x-5k)\cancel{(x-k)}}{(x-2k)\cancel{(x-k)}} = \dfrac{2(x-2k)-(x-5k)}{(x-2k)}$

$= \dfrac{2x-4k-x+5k}{(x-2k)} = \dfrac{x+k}{x-2k}$

b) $u = x+k \qquad v = x-2k \qquad \dfrac{vu'-uv'}{v^2}$

$u' = 1 \qquad v' = 1$

$\dfrac{(x-2k)-(x+k)}{(x-2k)^2} = \dfrac{-3k}{(x-2k)^2}$

If k is negative $-3k$ is +ve

$(x-2k)^2$ is +ve

$\therefore f'(x)$ is +ve

$\therefore$ Increasing function.